CHARLES UNIVERSITY
FACULTY OF MATHEMATICS AND PHYSICS
Richard Korytár
http://ctcm.kfkl.cz/statphys2020

Winter 2020

7.10.2020

Methods of Statistical Physics NFPL088, Wed 16:30, zoom/F155KFM

Problem Sheet 1

1 Fermi-Dirac distribution

Derive one-particle distribution function for systems of non-interacting fermionic particles. Reminder: the partition function is given by the formula

$$\mathcal{Z}(T,\mu) = \sum_{\{n_\lambda\}} \exp\left[\beta \sum_{\lambda=1}^{\mathcal{M}} n_\lambda(\mu - E_\lambda)\right], \quad (1)$$

where E_λ, n_λ are energies and occupations of single particle states.

2 Bose-Einstein distribution

Derive one-particle distribution function for systems of non-interacting bosonic particles, employing the Eq. (1).

3 Evaluation of observables by a source-term trick

The expectation value of an observable $\bar{A} = \langle \hat{A} \rangle$ in a quantum grand-canonical ensemble is given by the expression

$$\bar{A} = \frac{1}{\mathcal{Z}} \mathrm{Tr}\left[\hat{A} e^{-\beta(\hat{H}-\mu\hat{N})}\right]. \quad (2)$$

This expression can be evaluated alternatively by rewriting it as a derivative

$$\bar{A} = \beta^{-1} \left.\frac{\partial \ln \mathcal{Z}_\eta}{\partial \eta}\right|_{\eta=0}, \qquad \text{where } \mathcal{Z}_\eta = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N}-\eta\hat{A})}\right]. \quad (3)$$

Prove Eq.(3).

(a) Let's introduce an operator expression[1] $\exp\left(\hat{X} + \eta\hat{A}\right)$, defined by the Taylor series $\sum_{n=0}^{\infty}\left(\hat{X} + \eta\hat{A}\right)^n/n!$. Proceed to expand the Taylor series expression to linear order in η. To this end, justify the identity

$$\left(\hat{X} + \eta\hat{A}\right)^n = \hat{X}^n + \eta\left(\hat{A}\hat{X}^{n-1} + \hat{X}\hat{A}\hat{X}^{n-2} + \ldots + \hat{X}^{n-2}\hat{A}\hat{X} + \hat{X}^{n-1}\hat{A}\right) + \text{higher powers of } \eta.$$

(b) Under the trace the calculus simplifies considerably. Using the previous result, show that $\mathrm{Tr}\left[\exp\left(\hat{X} + \eta\hat{A}\right) - \exp\left(\hat{X}\right)\right] = \eta \mathrm{Tr}\left[\hat{A}\exp\left(\hat{X}\right)\right]$. Hint: operators under the trace can be permuted cyclically.

[1] For simplicity, you can think of $\hat{X}$ and $\hat{A}$ as square matrices, which do not commute, in general.

(c) Show, that this leads to Eq.(3).

4 **Curie's law of a classical moment**

Consider a classical magnetic moment $\boldsymbol{\mu} = \mu_0 \mathbf{e}$ of a fixed magnitude $\mu_0 = |\boldsymbol{\mu}|$, pointing in all possible directions $\mathbf{e}$, $|\mathbf{e}| = 1$. Under influence of an external magnetic field $\mathbf{b}$ the energy of the moment is $E(\boldsymbol{\mu}) = -\mathbf{b} \cdot \boldsymbol{\mu}$.

(a) Verify, that the average moment at temperature T and field $\mathbf{b} = (0, 0, b)$ reads

$$\bar{\boldsymbol{\mu}} = (0, 0, \bar{\mu}_z), \quad \bar{\mu}_z(T, b) \;=\; \mu_0 \, \mathcal{L}(\beta \mu_0 b), \tag{4}$$

where $\mathcal{L}(x) := \coth(x) - \frac{1}{x}$ is the Langevin's function.

(b) By performing (carefully) the limimit $b \to 0^+$, obtain the magnetic susceptibility $\kappa(T)$!

(c) Now calculate the susceptibility alternatively, using the formula which relates the linear response to fluctuations, given at the lecture. To this end, you will have to evaluate $\langle \mu_z \rangle_0(T)$ and $\langle \mu_z^2 \rangle_0(T)$.

(d) Compare with the result of (b).