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Methods of Statistical Physics
NFPL088, Wed 16:30, zoom/F155KFM

Problem Sheet 3

1 Classical gas in a gravitational field

Let's assume N classical point masses moving along a half-line $x > 0$ in a homogeneous field. The Hamilton's function is

$$H\Big(\{x_i\}_{i=1}^N,\, \{p_i\}_{i=1}^N\Big) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + c x_i\right), \tag{1}$$

where $x_i > 0$ are the coordinates, p_i the momenta and m is the mass ($c = mg$). For a classical canonical distribution at temperature T, derive

(a) the partition sum;

(b) the average particle density $\langle\rho(x)\rangle$, where $\rho(x) = \sum_i \delta(x - x_i)$. Verify, that your result gives the correct particle number, $\int \mathrm{d}x \langle\rho(x)\rangle = N$. Qualitatively discuss the temperature dependence.

(c) Calculate the internal energy. (Hint: it's not necessary to compute another integral.) Discuss the relation with equipartition theorem.

2 Model of an atom in a radiation field

We consider an atom which can be excited, either thermally or by the external field. For simplicity, we focus on the ground state and the first excited state only. Naturally, it is convenient to use Pauli matrices $\hat\sigma_i$, acting on the two atomic states. The Hamiltonian of the atom reads $\hat H_\mathrm{A} = \Delta\hat\sigma_z$. The Hamiltonian of the field, capable at inducing transitions, is $\hat H_\mathrm{F} = \varepsilon\hat\sigma_x$. Study the linear response to $\hat H_\mathrm{F}$ in the following steps:

(a) To familiarize yourself with the model, give the spectrum of $\hat H_\mathrm{A}$, the energetic gap and the probability of finding the atom in an excited state at a given temperature.

(b) Compute the susceptibility $\kappa_{AB}(T)$ with $\hat A = -\hat B = \hat\sigma_x$ following the definition given in the lecture [Eq. (41) therein].

(c) For $\Delta \neq 0$, evaluate the limits $T = 0$ and ∞ of the susceptibility. Then, evaluate the limit $\Delta = 0$ for any T. If the ground state is degenerate, the zero-temperature susceptibility should diverge.

(d) Compute the susceptibility $\kappa_{AB}(T)$ with $\hat A = \hat\sigma_x, \hat B = \hat\sigma_z$. Discuss the result: what does it imply?

Further reading: Quantum optics often employs the so called Jaynes–Cummings model, where $\hat{H}_\mathrm{A}$ is used to model an atom. The field is quantized, unlike in this problem, where it is classical and static.

3 Curie-like temperature behavior

Using the definition of the susceptibility [Eq. (41) from the lecture] show, that the susceptibility of a non-degenerate system is finite at low temperatures, and diverges if the system has a ground-state degeneracy. Discuss the examples: magnetic susceptibility of a Fermi gas and of a magnetic moment.