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Methods of Statistical Physics
NFPL088, Wed 16:30, zoom/F155KFM

Problem Sheet 5

1 Density of states of a paramagnet

Consider a paramagnet consisting of N atoms. Assume that it is sufficient to characterize each atom by a spin that can have only two values: ± 1 or "up" ↑ and "down" ↓. The total magnetization in dimensionless units is simply the number of up-spins minus the number of down-spins.

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(illustration)

(a) What is the total number of microstates that can be represented in this way?

(b) Let $N = N_+ + N_-$ where $N_\pm$ count the spins up (down). How many microstates yield given N_+, N_-? Hint: combination number.

(c) Using your last two results, give the probability $w(M)$ of the magnet having a magnetization M, assuming that all microstates have the same probability. Discuss the physical conditions to realize equal probabilities. When does this model apply to a real magnet?

(d) *Sanity check:* Convince yourself, that the probabilities of all macrostates $M = -N, -N + 2, \ldots, N-2, N$ sum to unity, *i.e.*

$$\sum_{M=\pm N, \pm(N-2), \ldots, 0} w(M) = 1.$$

Hint: Use binomial theorem for $(1+1)^N$. What is the mean magnetization of the magnet?

(e) Draw a plot of $w(M)$, knowing a few important values, *e.g.* $w(0)$ and $w(-N)$.

2 Paramagnet in the thermodynamic limit

We embark from the result of the previous problem to achieve a transparent result in the limit of (macroscopically) large N. Simplify the expression for $w(M)$ in the limit of large N and a small magnetization per atom, $m = M/N$, $m \ll 1$. To do so:

(a) Apply Stirling's formula $N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$ to all factorials. You get a product of terms with different powers and bases. Group terms with the same base together and arrive at the expression of the form $2^\alpha \pi^\beta N^\gamma (1+m)^\delta (1-m)^\epsilon$.

(b) For further convenience, evaluate the natural logarithm $\log w(mN)$.

(c) Taylor-expand the logarithm to second order in the small parameter m using

$$\log(1 \pm m) = \pm m - \frac{m^2}{2} + \mathcal{O}(m^2).$$

(d) Exponentiate to get $w(M) \approx \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{M^2}{2N}\right)$. Remember, you can discard small terms, *e.g.* $N + 1 + m \approx N$.

(e) Verify, that $\int_{-\infty}^{\infty} w(M)\mathrm{d}M = 2$. Why did we get 2 instead of 1 on the right side?

(f) With this approximate probability $w(M)$, calculate again the mean magnetization, $\langle M \rangle$ and recover your earlier result.

(g) Statistical fluctuations around the mean are expressed by the variance $\langle M^2 \rangle - \langle M \rangle^2$. How does the standard deviation (dispersion of the measured magnetizations), $\sqrt{\langle M^2 \rangle - \langle M \rangle^2}$, grow relative to the sample size? Discuss your result in terms of the central limit theorem.