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Methods of Statistical Physics
NFPL088, Wed 16:30, zoom/F155KFM

Problem Sheet 7

1 Quantum correction to the specific heat of an ideal atomic gas

Start with the expression for the internal energy of ideal fermions (bosons), $U = \sum_k \varepsilon_k/(e^{\beta(\varepsilon_k-\mu)} \pm 1)$.

(a) Perform the continuum limit and proove the exact relation $U = -\frac{3}{2}\Phi$ (see Eq. (11) from the lecture).

(b) Calculate the specific heat ($c_V = (\partial U/\partial T)_{V,N}$) in the limit of high temperatures using the fugacity expansion; give the quantum correction, which has the lowest nonzero order of $\hbar$.

(c) Discuss your result: what do you learn from it?

Remark: Alternative calculation proceeds via the entropy $S = -(\partial\Phi/\partial T)_{\mu,V}$ and $c_V = T\,(\partial S/\partial T)$. Note, however, that the chemical potential must be fixed.

2 Variance of particles in the BEC

The number of particles in the condensate is $\langle n_{\mathbf{p}=0}\rangle$. Calculate the variance of the number of particles in the condensate, $(\Delta n_{\mathbf{p}=0})^2 = \left\langle (n_{\mathbf{p}=0} - \langle n_{\mathbf{p}=0}\rangle)^2\right\rangle = \langle n^2_{\mathbf{p}=0}\rangle - \langle n_{\mathbf{p}=0}\rangle^2$ for a BEC under the critical temperature!

(a) Look into the Problem 2, Sheet 1. Use the same technique to calculate the variance of the number of particles in a state $\mathbf{p}$, $n_{\mathbf{p}}$. Prove the thermodynamic relation

$$(\Delta n_{\mathbf{p}=0})^2 = \beta^{-1}\left(\frac{\partial\langle n_{\mathbf{p}=0}\rangle}{\partial\mu}\right)_{T,V}. \quad (1)$$

(b) Express the ground-state occupation through the fugacity z,

$$\langle n_{\mathbf{p}=0}\rangle = \frac{z}{1-z}. \quad (2)$$

Use this expression and Eq. (1) to show, that $(\Delta n_{\mathbf{p}=0})^2 = \langle N\rangle + \langle N\rangle^2$. Discuss the result.

3 Bosons that do not condense

Show, that bosons with a quadratic dispersion, moving in two dimensions, do not undergo BEC. Use the continuum expression for the particle number, $N = \int \mathrm{d}\epsilon\, N(\epsilon)/(e^{\beta\epsilon}/z - 1)$. Unlike in the 3D case, the density of states in 2D is constant; for a square of length L it is easy to get $N(\epsilon) = L^2/\delta E$ for $\epsilon > 0$ and zero otherwise (The physical meaning of the constant δE is the average spacing between eigenenergies per unit area.). Now the integral for N can be calculated exactly. For fixed N and L^2, you obtain a relationship between the fugacity and the temperature, allowing you to express the temperature dependence of the chemical potential. Inspect the low- and high-T limits and sketch a graph of $\mu(T)$ at a fixed density N/L^2 (no computer!). Discuss the result.

4 Density of states of free particles: boundary effects

Consider free particles with a parabolic dispersion $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2m$ in a three-dimensional box of length L. Calculate the density of states using

(a) periodic boundary conditions

$$\psi(x+L, y, z) = \psi(x, y, z)$$

(and similar conditions in the y and z directions);

(b) hard-wall boundary conditions

$$\psi(0, y, z) = \psi(L, y, z) = 0$$

(and similarly in the y and z directions).

Show, that the single-particle ground-state $\psi^{(E=0)}(x, y, z)$ differs between cases (**a**) and (**b**), although the densities of states are equal.

5 Density of states of free particles: two dimensions

Show, that the density of states in two dimensions is $N(\epsilon) = \Theta(\varepsilon) \times L^2/\delta E$, where $\Theta(\varepsilon)$ is the Heaviside function and δE is the difference between the first-excited and ground-state energies.