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Methods of Statistical Physics
NFPL088, Tue 10:30, F052 (KFKL)

Problem Sheet 2

1 Classical gas in a gravitational field

Let's assume N classical point masses moving along a half-line $x > 0$ subject to a constant gravitational force, mg. The Hamilton's function is

$$H\Big(\{x_i\}_{i=1}^N,\, \{p_i\}_{i=1}^N\Big) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + cx_i\right), \tag{1}$$

where $x_i > 0$ are the coordinates, p_i the momenta and m is the mass ($c = mg$). For a classical canonical distribution at temperature T, derive

- **(a)** the partition sum;
- **(b)** the average particle density $\langle\rho(x)\rangle$, where $\rho(x) = \sum_i \delta(x - x_i)$. Verify, that your result gives the correct particle number, $\int \mathrm{d}x \langle\rho(x)\rangle = N$. Qualitatively discuss the temperature dependence.
- **(c)** Calculate the internal energy. (Hint: it's not necessary to compute another integral.) Discuss the relation with equipartition theorem.

2 Density of states of free particles: two dimensions

Show, that the density of states in two dimensions is $N(\epsilon) = \Theta(\varepsilon) \times L^2/\delta E$, where $\Theta(\varepsilon)$ is the Heaviside function and δE is the difference between the first-excited and ground-state energies.

3 Fluctuation-dissipation relation: classics

Prove the relation

$$(\Delta H)^2\,(T) = k_{\mathrm{B}} T^2\, C(T) \tag{2}$$

in classical statistics, using the definitions of $F(T), S(T)$ and $Z(T)$ given in the lecture.

4 Fluctuation-dissipation relation: quantum statistics

Prove the relation

$$(\Delta H)^2\,(T) = k_{\mathrm{B}} T^2\, C(T) \tag{3}$$

in quantum statistics, using the definitions of $F(T), S(T)$ and $Z(T)$ given in the lecture.

5 **Response function**

Proove the relationship given in the lecture:

$$-Z^{-1}(T,0)\times\ \mathrm{Tr}\left[A\exp(-\beta H_0)\int_0^\beta \exp(\alpha H_0)\,B\exp(-\alpha H_0)\,\mathrm{d}\alpha\right]=\sum_{mn}A_{mn}B_{nm}\,\frac{w_m(T)-w_n(T)}{E_m-E_n}$$

Hint: evaluate the operator products in the orthonormal basis.

6 **Statistical mechanics of neutron beams**

Suppose that a neutron beam is prepared in the following way: half of the neutrons have their spin aligned with the (positive) x-axis and the other half along the z-axis.

(a) Define the appropriate density operator $\hat{\rho}$ for the beam. Verify that $\hat{\rho}$ satisfies the three basic properties: hermicity, positivity and unitarity. Does the beam represent a pure state?

(b) Calculate the average spin vector $\langle \mathbf{s}\rangle = \frac{1}{2}\hbar\langle\hat{\boldsymbol{\sigma}}\rangle$ and the corresponding variances $\langle \hat{s}_i^2\rangle - \langle s_i\rangle^2$, $i = x, y, z$. Discuss your results.

(c) Calculate the eigenvalues P_+, P_- of the density operator and the so called degree of polarization defined through

$$\Pi := \frac{P_+ - P_-}{P_+ + P_-}. \tag{4}$$

How do you interpret your result?