

Methods of Statistical Physics
NFPL088, Tue 10:30, F052 (KFKL)

Problem Sheet 3

1 Fermi-Dirac distribution

Derive one-particle distribution function for systems of non-interacting fermionic particles. Reminder: the partition function is given by the formula

$$\mathcal{Z}(T, \mu) = \sum_{\{n_\lambda\}} \exp \left[\beta \sum_{\lambda=1}^{\mathcal{M}} n_\lambda (\mu - E_\lambda) \right], \quad (1)$$

where E_λ, n_λ are energies and occupations of single particle states.

2 Bose-Einstein distribution

Derive one-particle distribution function for systems of non-interacting bosonic particles, employing the Eq. (1).

3 Evaluation of observables by a source-term trick

The expectation value of an observable $\bar{A} = \langle \hat{A} \rangle$ in a quantum grand-canonical ensemble is given by the expression

$$\bar{A} = \frac{1}{\mathcal{Z}} \text{Tr} \left[\hat{A} e^{-\beta(\hat{H} - \mu \hat{N})} \right]. \quad (2)$$

This expression can be evaluated alternatively by rewriting it as a derivative

$$\bar{A} = \beta^{-1} \left. \frac{\partial \ln \mathcal{Z}_\eta}{\partial \eta} \right|_{\eta=0}, \quad \text{where } \mathcal{Z}_\eta = \text{Tr} \left[e^{-\beta(\hat{H} - \mu \hat{N} - \eta \hat{A})} \right]. \quad (3)$$

Prove Eq.(3).

(a) Let's introduce an operator expression¹ $\exp(\hat{X} + \eta \hat{A})$, defined by the Taylor series

$\sum_{n=0}^{\infty} (\hat{X} + \eta \hat{A})^n / n!$. Proceed to expand the Taylor series expression to linear order in η . To this end, justify the identity

$$(\hat{X} + \eta \hat{A})^n = \hat{X}^n + \eta \left(\hat{A} \hat{X}^{n-1} + \hat{X} \hat{A} \hat{X}^{n-2} + \dots + \hat{X}^{n-2} \hat{A} \hat{X} + \hat{X}^{n-1} \hat{A} \right) + \text{higher powers of } \eta.$$

¹For simplicity, you can think of \hat{X} and \hat{A} as square matrices, which do not commute, in general.

- (b) Under the trace the calculus simplifies considerably. Using the previous result, show that to linear order $\text{Tr} \left[\exp \left(\hat{X} + \eta \hat{A} \right) - \exp \left(\hat{X} \right) \right] = \eta \text{Tr} \left[\hat{A} \exp \left(\hat{X} \right) \right]$. Hint: operators under the trace can be permuted cyclically.
- (c) Show, that this leads to Eq.(3).

4 Curie's law of a classical moment

Consider a classical magnetic moment $\boldsymbol{\mu} = \mu_0 \mathbf{e}$ of a fixed magnitude $\mu_0 = |\boldsymbol{\mu}|$, pointing in all possible directions \mathbf{e} , $|\mathbf{e}| = 1$. Under influence of an external magnetic field \mathbf{b} the energy of the moment is $E(\boldsymbol{\mu}) = -\mathbf{b} \cdot \boldsymbol{\mu}$.

- (a) Verify, that the average moment at temperature T and field $\mathbf{b} = (0, 0, b)$ reads

$$\bar{\boldsymbol{\mu}} = (0, 0, \bar{\mu}_z), \quad \bar{\mu}_z(T, b) = \mu_0 \mathcal{L}(\beta \mu_0 b), \quad (4)$$

where $\mathcal{L}(x) := \coth(x) - \frac{1}{x}$ is the Langevin's function.

- (b) By performing (carefully) the limit $b \rightarrow 0^+$, obtain the magnetic susceptibility $\kappa(T)$!
- (c) Now calculate the susceptibility alternatively, using the formula which relates the linear response to fluctuations, given at the lecture. To this end, you will have to evaluate $\langle \mu_z \rangle_0(T)$ and $\langle \mu_z^2 \rangle_0(T)$.
- (d) Compare with the result of (b).

5 Fluctuations of particle number in the grandcanonical ensemble

Prove the formula given at the lecture,

$$(\Delta N)^2(T, \mu) = k_B T \frac{\partial N(T, \mu)}{\partial \mu}. \quad (5)$$

Choose either classical or quantum statistics. Hint: proceed as in the Problem Sheet 2, problems **3**, **4**.