CHARLES UNIVERSITY
FACULTY OF MATHEMATICS AND PHYSICS
Richard Korytár
http://ctcm.kfkl.cz/statphys2021

Winter 2021

9.11.2021

Methods of Statistical Physics
NFPL088, Tue 11:30, F052 (KFKL)

Problem Sheet 4

Literature: any decent Statistical Mechanics textbook

1 Spin susceptibility of free electrons

Relativistic quantum field theory predicts that electrons carry a spin $S = 1/2$. Associated with the spin is a magnetic dipole moment, which can have values $+\mu_0$, $-\mu_0$ for the spin states $\uparrow$, $\downarrow$. Consider a non-interacting electron gas with single-particle energies $\epsilon_{\lambda,\uparrow} = \epsilon_{\lambda,\downarrow}$, equal for both spin directions in absence of a magnetic field (Think why!). Now we assume that a small magnetic field $\mathbf{B} = (0, 0, B)$ is switched on. Furthermore, we assume that this field interacts only with the magnetic dipole moment of the spins, not with the orbital motion of the electrons. This means we neglect the Lorentz force.

(a) What is the change in the single-particle eigenvalues $\epsilon_{\lambda,\sigma}$, $\sigma =\uparrow, \downarrow$, after B is switched on?

(b) Motivate the definition of the magnetization's

$$M_z = \mu_0 \sum_{\lambda} \left[n_{\mathrm{F}}(\epsilon_{\lambda,\uparrow} - \mu) - n_{\mathrm{F}}(\epsilon_{\lambda,\downarrow} - \mu) \right], \tag{1}$$

for non-interacting particles, where n_{F} denotes the Fermi function and μ (not μ_0) is the chemical potential.

(c) Dimensional analysis: The spin susceptibility $\chi = \frac{\partial M}{\partial B}\big|_{B=0}$ describes how the magnetization reacts to an external field. Argue, that at zero temperature, the susceptibility has to be proportional to μ_0^2/E, where E has the dimension of energy. List a few candidates for the (until now unknown) quantity E (cohesive energy, Debye energy etc.). Think about the only *reasonable* correct answer!

(d) Calculate M_z in leading order of B and in the limit of $T \to 0$. Derive the zero-temperature limit of χ!

(e) *Discussion, but no calculation!* Review the leading correction from the Sommerfeld expansion and discuss the terms from the view of the dimension analysis. For typical metals, are these corrections small/large and why?

(f) *Optional:* The above calculation tacitly ignored the spin-orbit coupling. What would change in the formula (1) if we doing it right?

Turn the page!

In the next two problems, we shall explore the temperature dependencies of observables in an ideal Bose gas of spin-less particles. Use the continuum form of the grand-canonical potential

$$\Omega = kT \int_0^\infty N(\epsilon) \ln \left[1 - e^{-(\epsilon-\mu)\beta}\right] \mathrm{d}\epsilon \tag{2}$$

where $N(\epsilon)$ denotes the (single particle) density of states.

We will assume two forms of $N(\epsilon)$:

- For particles with a quadratic dispersion (in three dimensions) $N(\epsilon) = Vb\sqrt{\epsilon}$ (it can be calculated easily); V is the volume and b a constant. Examples: atoms in a gas, ferromagnetic magnons.
- For particles with a linear dispersion; $N(\epsilon) = Vb'\epsilon^2$. Examples: acoustic phonons, photons, antiferromagnetic magnons.

2 Thermodynamics of free bosons: parabolic dispersion

(a) Plot the Bose-Einstein distribution function for a finite T. What are the allowed values of μ?

(b) Certain class of bosons has a vanishing chemical potential. These are phonons, but also atoms in a Bose-Einstein condensation (BEC), to be discussed in detail at the end of the semester. Convince yourself, that the integrand [Eq. (2)] is well defined in the limit $\epsilon \to 0^+$ when $\mu = 0$. Show, that the grand-canonical potential can be written in the form $\Omega = aVT^{5/2}$, where a is independent of T.

(c) Using the previous result, give the temperature dependence of the entropy and heat capacity $c_V(T)$. Show, that the pressure ($P = -\Omega/V$) is volume-independent.

Remark: this is the temperature dependence of ferromagnetic magnons, for example

3 Thermodynamics of free bosons: linear dispersion

The calculation of thermodynamic quantities for a linear dispersion proceeds analogously as with the parabolic dispersion.

(a) Repeat the previous problem to verify, that $c_V(T) \propto T^3$ for a linear dispersion. Discuss: Lattice contribution to the specific heat of a solid.

(b) For $\mu = 0$, it holds that $\Omega = U - TS$, where U is the internal energy. Show, that $U(T) \propto T^4$. Discuss the Stefan-Boltzmann law.