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Methods of Statistical Physics
NFPL088, Tue 17:20, F155 (KFM)

Problem Sheet 1

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1 Thermodynamics of a classical magnetic moment in the magnetic field
Consider a single classical magnetic moment $\boldsymbol{\mu}$ in an external magnetic field $\mathbf{B}$ at a temperature T.

(a) Calculate the average value of $\boldsymbol{\mu}$ in the limit of small fields and high temperatures.

(b) Calculate the magnetic susceptibility.

2 Thermodynamics of a quantum spin in the magnetic field
Repeat the previous problem with a quantum spin $\hat{\mathbf{S}}$ with a fixed square, $\hat{\mathbf{S}}^2 = S(S+1)$ (*e.g.* an electron or certain atoms). The Hamiltonian describing the free spin in a field is $\hat{H} = g\mu_\mathrm{B}\mathbf{B}\cdot\hat{\mathbf{S}}$.

3 Classical gas in a gravitational field
Let's assume N classical point masses moving along a half-line $x > 0$ subject to a constant gravitational force, mg. The Hamilton's function is

$$H\Big(\{x_i\}_{i=1}^N,\,\{p_i\}_{i=1}^N\Big) = \sum_{i=1}^N \left(\frac{p_i^2}{2m} + cx_i\right), \tag{1}$$

where $x_i > 0$ are the coordinates, p_i the momenta and m is the mass ($c = mg$). For a classical canonical distribution at temperature T, derive

(a) the partition sum;

(b) the average particle density $\langle\rho(x)\rangle$, where $\rho(x) = \sum_i \delta(x - x_i)$. Verify, that your result gives the correct particle number, $\int \mathrm{d}x\langle\rho(x)\rangle = N$. Qualitatively discuss the temperature dependence.

(c) Calculate the internal energy. (Hint: it's not necessary to compute another integral.) Discuss the relation with equipartition theorem.

4 Density of states of free particles: boundary effects
Consider free particles with a parabolic dispersion $\varepsilon_\mathbf{p} = \mathbf{p}^2/2m$ in a three-dimensional box of length L. Calculate the density of states using

(a) periodic boundary conditions

$$\psi(x+L, y, z) = \psi(x, y, z)$$

(and similar conditions in the y and z directions);

(b) hard-wall boundary conditions

$$\psi(0, y, z) = \psi(L, y, z) = 0$$

(and similarly in the y and z directions).

Show, that the single-particle ground-state $\psi^{(E=0)}(x, y, z)$ differs between cases (**a**) and (**b**), although the densities of states are equal.

5 **Density of states of free particles: two dimensions**
Show, that the density of states in two dimensions is $N(\epsilon) = \Theta(\varepsilon) \times L^2/\delta E$, where $\Theta(\varepsilon)$ is the Heaviside function and δE is the difference between the first-excited and ground-state energies.

6 **Density of states of free particles: one dimension**
Derive the formula for the density of states in one dimension for a free "particle-in-a-box". Show, that the density of states has a singularity. Is the singularity integrable? In other words, is the number of states located in an arbitrary finite energy window finite?
Additional reading: van Hove singularities