

## Methods of Statistical Physics NFPL088, Tue 17:20, F155 (KFM)

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### Problem Sheet 5

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#### 1 Quantum correction to the specific heat of an ideal atomic gas

Start with the expression for the internal energy of ideal fermions (bosons),  $U = \sum_k \varepsilon_k / (e^{\beta(\varepsilon_k - \mu)} \pm 1)$ .

- (a) Perform the continuum limit and prove the exact relation  $U = -\frac{3}{2}\Phi$  (see the expression for  $\Phi$  from the lecture).
- (b) Calculate the specific heat ( $c_V = (\partial U / \partial T)_{V,N}$ ) in the limit of high temperatures using the fugacity expansion; give the quantum correction, which has the lowest nonzero order of  $\hbar$ .
- (c) Discuss your result: what do you learn from it?

*Remark:* Alternative calculation proceeds via the entropy  $S = -(\partial \Phi / \partial T)_{\mu,V}$  and  $c_V = T (\partial S / \partial T)$ . Note, however, that the chemical potential must be fixed when these derivatives are calculated.

#### 2 Variance of particles in the BEC

The number of particles in the condensate is  $\langle n_{\mathbf{p}=0} \rangle$ . Calculate the variance of the number of particles in the condensate,  $(\Delta n_{\mathbf{p}=0})^2 = \langle (n_{\mathbf{p}=0} - \langle n_{\mathbf{p}=0} \rangle)^2 \rangle = \langle n_{\mathbf{p}=0}^2 \rangle - \langle n_{\mathbf{p}=0} \rangle^2$  for a BEC under the critical temperature!

- (a) Look into the Problem [1](#), Sheet 3 (Fermi-Dirac). Use the same technique to calculate the variance of the number of particles in a state  $\mathbf{p}$ ,  $n_{\mathbf{p}}$ . Prove the thermodynamic relation

$$(\Delta n_{\mathbf{p}=0})^2 = \beta^{-1} \left( \frac{\partial \langle n_{\mathbf{p}=0} \rangle}{\partial \mu} \right)_{T,V}. \quad (1)$$

- (b) Express the ground-state occupation through the fugacity  $z$ ,

$$\langle n_{\mathbf{p}=0} \rangle = \frac{z}{1-z}. \quad (2)$$

Use this expression and Eq. (1) to show, that  $(\Delta n_{\mathbf{p}=0})^2 = \langle N \rangle + \langle N \rangle^2$ . Discuss the result.

#### 3 Bosons that do not condense

Show, that bosons with a quadratic dispersion, moving in two dimensions, do not undergo BEC. Use the continuum expression for the particle number,  $N = \int d\epsilon N(\epsilon) / (e^{\beta\epsilon} / z - 1)$ . Unlike in the 3D case, the density of states in 2D is constant; for a square of length  $L$  it is easy to get  $N(\epsilon) = L^2 / \delta E$  for  $\epsilon > 0$  and zero otherwise (The physical meaning of the constant  $\delta E$  is the average spacing between

eigenenergies per unit area.). Now the integral for  $N$  can be calculated exactly. For fixed  $N$  and  $L^2$ , you obtain a relationship between the fugacity and the temperature, allowing you to express the temperature dependence of the chemical potential. Inspect the low- and high- $T$  limits and sketch a graph of  $\mu(T)$  at a fixed density  $N/L^2$  (no computer!). Discuss the result.